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# Some properties of the linearized model of the (super) $p$ -brane

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## Abstract

Some general properties of the relativistic  $p$ -dimensional surface imbedded into  $D$ -dimensional spacetime and its reduction to the simplest case of the quadratic Lagrangian (the linearized model) are considered. The solutions of the equations of motion of the linearized model for the  $p$ -brane with arbitrary topology and massless eigenstates, as well as with critical dimension after quantization are presented. Some generalizations for the supermembrane are discussed.

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# 1 Introduction

Nowadays not only one-dimensional relativistic objects - strings, but also the objects of higher dimension -  $p$ -dimensional (super) $p$ -branes are suggested as substantial physical and mathematical objects. As for their properties, much less about those of  $p$ -branes is known this far [1-7].

The necessity to consider multidimensional objects with more than one space dimensions arises in various parts of the field theory. In particular, we may try to consider the (super) $p$ -brane theory as fundamental, like the (super)string theory ( $p = 1$ ) [8], as well as an effective model of supergravity, as shown in [9]. A possible correlation between ordinary and rigid (super) $p$ -branes and, in particular, the correlation between the rigid string and the ordinary membrane at  $p=2$  has been considered in [10, 11]. The calculation of the static potential for the  $p$ -brane compactified on the space-times of the various forms has been considered in [12, 13]

For the supermembrane ( $p = 2$ ), action is a direct multidimensional generalization of the string action [8]:

$$S = -\frac{T}{2} \int d^3\xi [\sqrt{h} h^{ij} \Pi_i^a \Pi_j^b \eta_{ab} - \sqrt{h} + 2\varepsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C B_{CBA}], \quad (1)$$

where  $T$  is the parameter of tension with the dimension  $(mass)^{(p+1)}$  or  $(length)^{-(p+1)}$ ,  $\xi^i$  ( $i = 0, 1, \dots, p$ ) are the worldvolume coordinates,  $h_{ij}$  is the metric of the worldvolume,  $h = -\det(h_{ij})$ ,  $\eta_{ab}$  is the Minkowski spacetime metric, and  $\Pi_i^A = \partial_i Z^M E_M^A$ ,  $A = a, \alpha$ ;  $M = \mu, \dot{\alpha}$ . Here,  $Z^M$  are the coordinates of the  $D$ -dimensional curved superspace, and  $E_M^A$  is the supervielbein. The 3-form  $B = \frac{1}{6} E^A E^B E^C B_{CBA}$ ,  $E^A = dZ^M E_M^A$  is the potential for the closed 4-form  $H = dB$ .

The action (1) is invariant respecting the global  $D$ -dimensional Poincaré transformations, as well as it is invariant respecting local parametrizations of the worldvolume with the parameters  $\eta^i(\xi)$ :

$$\delta Z^M = \eta^i(\xi) \partial_i Z^M, \quad \delta h_{ij} = \eta^k \partial_k h_{ij} + 2\partial_{(i} \eta^k h_{j)k}. \quad (2)$$

It is also invariant under local fermionic “ $k$ -transformations”:

$$\delta Z^M E_M^a = 0, \quad (3)$$

$$\delta Z^M E_M^\alpha = (1 + \Gamma)_\beta^\alpha k^\beta, \quad (4)$$

$$\begin{aligned} \delta(\sqrt{h}h^{ij}) = & -2i(1 + \Gamma)_\beta^\alpha k^\beta (\Gamma_{ab})_{\alpha\gamma} \Pi_n^\gamma h^{n(i} \varepsilon^{j)kl} \Pi_k^a \Pi_l^b - \\ & - \frac{2i}{3\sqrt{h}} k^\alpha (\Gamma_c)_{\alpha\beta} \Pi_k^\beta \Pi_l^c h^{kl} \varepsilon^{mn(i} \varepsilon^{j)pq} \times \\ & \times (\Pi_m^a \Pi_{pa} \Pi_n^b \Pi_{qb} + \Pi_m^a \Pi_{pa} h_{nq} + h_{mp} h_{nq}), \end{aligned} \quad (5)$$

with an anticommuting spacetime spinor  $k^\alpha(\xi)$ , and the matrix  $\Gamma$  defined by

$$\Gamma = \frac{1}{6\sqrt{h}} \varepsilon^{ijk} \Pi_i^a \Pi_b^j \Pi_k^c \Gamma_{abc} . \quad (6)$$

Unlike the two-dimensional string action, the action (1) at  $p \neq 1$  is not invariant respecting local conformal transformations with the parameter  $\Lambda(\xi)$ :

$$\delta Z^M = 0; \quad (7)$$

$$\delta h^{\alpha\beta} = \Lambda(\xi) h^{\alpha\beta} . \quad (8)$$

Varying the initial action leads to essentially non-linear field equations

$$\begin{aligned} \partial_i(\sqrt{h}h^{ij}\Pi_j^a) + \sqrt{h}h^{ij}\Pi_j^b \Pi_i^C \Omega_{Cb}^a + i\varepsilon^{ijk}\Pi_{ib}(\Pi_j^\alpha \Gamma_{\alpha\beta}^{ab} \Pi_k^\beta) + \\ + \varepsilon_{ijk}\Pi_i^b \Pi_j^c \Pi_k^d H_{bcd}^a = 0, \end{aligned} \quad (9)$$

$$[(1 - \Gamma)h^{ij}\Pi_i^\mu \Gamma_\mu]_\beta^\alpha \Pi_j^\beta = 0, \quad (10)$$

where  $\Omega_B^A$  is the 1-form connection in the  $D$ -dimensional curved superspace, and to the “embedding” equation

$$h_{ij} = \Pi_i^a \Pi_j^b \eta_{ab} , \quad (11)$$

which remains non-linear at any gauge. Their solution is known for certain simplest cases [1].

For open membranes, or for the existing open dimensions, at  $\sigma_i = \sigma_i^a, \sigma_i = \sigma_i^b$  the border condition is observed on the coordinates  $Z^M(\xi)$ :

$$\int d^3\xi \partial_i (\delta Z^a \sqrt{h} h^{ij} \Pi_{ja} + 3\varepsilon^{ijk} \delta Z^A \Pi_j^B \Pi_k^C B_{CBA}) = 0, \quad (12)$$

where  $h_{ij}$  is given by equation (11).

Any new solution of the equations of motion (9) and (10) describing the motion of a multidimensional relativistic object, on one hand, is of interest in itself, and on the other hand, it serves as a starting point for semiclassical quantization, when the minor variations respecting the known classical solution are investigated.

We have considered a mathematically simpler case at  $p=2$ . M.Duff in [3] presents a  $p$ -dimensional generalization of the supermembrane action, which has similar properties.

In the case when a complicated, non-linear dynamic system is investigated, it seems reasonable to start from its linearized model. This work aims to investigate a special type of action corresponding to a linearized model of the relativistic (super) $p$ -brane. Such approach is possible in all cases when the (super) $p$ -brane model appears.

## 2 A linearized model of the bosonic $p$ -brane

Let us consider as a less complicated the case of the bosonic relativistic  $p$ -brane. This means that we are considering the action

$$S = -T \int d^{p+1}\xi |\det(\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu})|^{\frac{1}{2}}, \quad (13)$$

where  $\xi = (\tau, \sigma_1, \dots, \sigma_p)$ ,  $\xi_\alpha \in [\xi_\alpha^a, \xi_\alpha^b]$ ,  $X^\mu = X^\mu(\tau, \sigma_1, \dots, \sigma_p)$ ,  $\mu = 0, \dots, D-1$ , where  $D$  is the dimension of the Minkowski spacetime with the metric  $g_{\mu\nu}$ ;  $\alpha = 0, \dots, p$ , where  $p$  is the space dimension of  $p$ -brane.

The equation of motion

$$\partial_\alpha (\sqrt{h} h^{\alpha\beta} \partial_\beta X^\mu) = 0, \quad (14)$$

resulting from (13), in the case the border conditions are taken into account, may be obtained from the classically equivalent action

$$S = -\frac{T}{2} \int d^{p+1}\xi \sqrt{h} [h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} - (p-1)] , \quad (15)$$

where an auxiliary metric  $h_{\alpha\beta}$  on the worldvolume of the membrane is introduced. The actions (13) and (15) to be equivalent, the metric  $h_{\alpha\beta}$  must obeys the imbedding condition:

$$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} , \quad (16)$$

like the embedding condition (11) in the supersymmetric case.

Besides, we must check if the constraint conditions  $p+1$  are observed:

$$P_\tau^\mu X_{\mu;i} = 0, \quad P^2 + T^2 \text{deth}_{ij} = 0, \quad (17)$$

where  $P_\tau^\mu = \delta\mathcal{L}/\delta\dot{X}^\mu$ ,  $1 \leq i, j \leq p$ .

We cannot quantize action (15) at  $p > 1$ , but we can introduce a certain simplification. Let  $Y^\mu$  be a variation respecting the classical solution  $X_0^\mu$ :

$$X^\mu = X_0^\mu + \varepsilon Y^\mu . \quad (18)$$

Then the equation of motion (14) turns into

$$\partial_\alpha P^{\mu\alpha} = \partial_\alpha P_0^{\mu\alpha} + \varepsilon \partial_\alpha C^{\mu\alpha} + o(\varepsilon) = 0 . \quad (19)$$

The requirement of the  $X^\mu$ -solution of the equation of motion being the first order in  $\varepsilon$  leads to the equation  $\partial_\alpha C^{\mu\alpha} = 0$ :

$$\partial_\alpha \frac{\partial A}{\partial X_{\mu;\alpha}} + \frac{3}{2h^0} \sum_{\alpha=0}^p \partial_\alpha \left( A \frac{\partial h^0}{\partial \dot{X}_{0\mu}} \right) = 0, \quad (20)$$

where  $A = \sum_{i,j=0}^p \partial_i X_0^\mu \partial_j Y_\mu \bar{h}_{ij}^0$ ,  $h_{ij}^0 = \partial_i X_0^\mu \partial_j X_{0\mu}$ ,  $h^0 = \text{deth}_{ij}^0$ .

The exact expression for the equation of motion (20) depends on the solution  $X_0^\mu(\xi)$ . For instance we may consider special tipe of the solution with

one or few compactified dimensions. The solution for the toroidal membrane on the spacetime with the topology  $R^{D-2} \times S^1 \times S^1$  is

$$X^1 = l_1 R_1 \sigma, \quad X^2 = l_2 R_2 \rho, \quad X^I = 0, \quad I = 3, \dots, D, \quad (21)$$

where  $0 \leq \sigma \leq 2\pi$ ,  $0 \leq \rho \leq 2\pi$ ,  $R_1$  and  $R_2$  are the radii of the two circles, and  $l_1$  and  $l_2$  are the integers characterizing the winding numbers of the membrane around the two circles.

In the light cone gauge,  $X^+ = p^+ \tau$ . The worldvolume metric on this background is flat,

$$g_{ij} = \text{diag}(-(l_1 l_2 R_1 R_2)^2, (l_1 R_1)^2, (l_2 R_2)^2), \quad (22)$$

and  $X^-$  is

$$X^- = \frac{1}{2p^+} (l_1 l_2 R_1 R_2)^2 \tau. \quad (23)$$

If we consider the fluctuations  $Z^\mu$  of the transverse coordinate around this classical solution

$$X^1 = \sigma + Z^1, \quad X^2 = \rho + Z^2, \quad X^I = Z^I, \quad I = 3, \dots, D, \quad (24)$$

then, keeping only the terms of the linear order in  $Z$ , we find

$$\ddot{Z}^1 = \partial_\sigma \partial_\sigma Z^1 + \partial_\sigma \partial_\rho Z^2, \quad \ddot{Z}^2 = \partial_\rho \partial_\rho Z^2 + \partial_\sigma \partial_\rho Z^1, \quad \ddot{Z}^I = \partial_\sigma \partial_\sigma Z^I + \partial_\rho \partial_\rho Z^I, \quad (25)$$

We may fix the remaining gauge invariance. The gauge choice  $g_{0\alpha} = 0$  can be solved for  $\partial_a X^-$ . Upon linearization on our background, this constraint gives

$$\partial_\rho \dot{Z}^1 = \partial_\sigma \dot{Z}^2, \quad (26)$$

from which follows the possibility

$$\partial_\rho Z^1 = \partial_\sigma Z^2. \quad (27)$$

This allows us to rewrite (24) in the form of the standard wave equations:

$$\ddot{Z}^1 = \partial_\sigma \partial_\sigma Z^1 + \partial_\rho \partial_\rho Z^1, \quad \ddot{Z}^2 = \partial_\sigma \partial_\sigma Z^2 + \partial_\rho \partial_\rho Z^2, \quad (28)$$

$$\ddot{Z}^I = \partial_\sigma \partial_\sigma Z^I + \partial_\rho \partial_\rho Z^I. \quad (29)$$

Equations of motion (28) and (29) are a special case of the equations (20). But here it should be noted that, as follows from (22) and (26), there is a special gauge condition, in which the general equation (20) turns into the ordinary wave equation.

The way described above is the investigation of small variations considering the classical solution. We may as well try to investigate the original action (13).

Let us introduce new variables  $\bar{X}^\mu$ :

$$\partial^\alpha \bar{X}^\mu = \sqrt{|h|} h^{\alpha\beta} \partial_\beta X^\mu . \quad (30)$$

This means that

$$\bar{h} = \det(\partial_\alpha \bar{X}^\mu \partial_\beta \bar{X}_\mu) = \text{sign}(h) |h|^{(p+1)^2+1}. \quad (31)$$

With these variables, the equation of motion (14) turns into the wave equation

$$\partial_\alpha \partial^\alpha \bar{X}^\mu = 0 , \quad (32)$$

and the conditions of the constraints (17) turn into

$$\bar{P}^2 + T^2 |\bar{h}|^{-\frac{p^2}{(p+1)^2+1}} \det(\partial_i \bar{X}^\mu \partial_j \bar{X}_\mu) = 0, \quad (33)$$

where  $\bar{P}^\mu \equiv \dot{\bar{X}}^\mu$  and  $i, j = 1, \dots, p$  are space indexes of the membrane.

For the sake of convenience, the space parameter of the membrane  $\xi_i \in [\xi_i^a; \xi_i^b]$  is considered  $\sigma_i \in [0; \pi]$  and for the open dimension

$$X^\mu(\tau, \dots, \sigma_i = 0, \dots) \neq X^\mu(\tau, \dots, \sigma_i = \pi, \dots), \quad (34)$$

unlike for the closed dimension, where the condition of periodicity is observed:

$$X^\mu(\tau, \dots, \sigma_i, \dots) = X^\mu(\tau, \dots, \sigma_i + \pi, \dots) , \quad (35)$$

or

$$X^\mu(\tau, \dots, \sigma_i, \dots) = X^\mu(\tau, \dots, \sigma_i + 2\pi, \dots) , \quad (36)$$

depending on the spheric or toroidal type of compactification.

The border conditions for the  $p$ -brane in bar variables  $\bar{X}^\mu$  are the same like ordinary variables  $X^\mu$ . If we can the motion of the  $p$ -brane express in  $X^\mu$  variables with the equation of motion (32), then the solution of this equation may be written.

In the general case, for the membrane with an arbitrary topology, when there are  $p_0$  open dimensions,  $p_1$  closed dimensions with the period  $\pi$ , and  $p_2$  closed dimensions with the period  $2\pi$  ( $p_0 + p_1 + p_2 = p$ ), the solution of equation (32) may be as follows:

$$\begin{aligned}
X^\mu(\xi) = & X^\mu + \frac{1}{\pi^p T} p^\mu \tau + \\
& + i \sqrt{\frac{2^{p-1}}{\pi^p T}} \sum_{\mathbf{n}} n^{-1} (\alpha_{\mathbf{n}}^\mu e^{-in\tau} - \alpha_{\mathbf{n}}^{*\mu} e^{in\tau}) \prod_{i=1}^p \cos n_i \sigma_i + \\
& + i \sqrt{\frac{2^{p-1}}{\pi^p T}} \sum_{\mathbf{m}} m^{-1} [(\alpha_{\mathbf{m}}^\mu e^{-2im\tau} - \alpha_{\mathbf{m}}^{*\mu} e^{2im\tau}) e^{-2i\bar{m}\bar{\sigma}} + \\
& \quad + (\beta_{\mathbf{m}}^\mu e^{-2im\tau} - \beta_{\mathbf{m}}^{*\mu} e^{2im\tau}) e^{2i\bar{m}\bar{\sigma}}] + \\
& + i \sqrt{\frac{2^{p-1}}{\pi^p T}} \sum_{\mathbf{k}} k^{-1} [(\alpha_{\mathbf{k}}^\mu e^{-ik\tau} - \alpha_{\mathbf{k}}^{*\mu} e^{ik\tau}) e^{-i\bar{k}\bar{\sigma}} + \\
& \quad + (\beta_{\mathbf{k}}^\mu e^{-ik\tau} - \beta_{\mathbf{k}}^{*\mu} e^{ik\tau}) e^{i\bar{k}\bar{\sigma}}],
\end{aligned} \tag{37}$$

where  $X^\mu$  are the initial coordinates of the mass centrum and  $p^\mu$  is the impuls of the mass centrum of the membrane at

$$\begin{aligned}
\mathbf{n} & \in \mathbf{N}^{p_0} \setminus 0, \quad n = \sqrt{n_1^2 + \dots + n_{p_0}^2} ; \\
\mathbf{m} & \in \mathbf{N}^{p_1} \setminus 0, \quad m = \sqrt{m_1^2 + \dots + m_{p_1}^2} ; \\
\mathbf{k} & \in \mathbf{N}^{p_2} \setminus 0, \quad k = \sqrt{k_1^2 + \dots + k_{p_2}^2} ; \\
\bar{m}\bar{\sigma} & \equiv m_{p_0+1} \sigma_{p_0+1} + \dots + m_{p_0+p_1} \sigma_{p_0+p_1} , \\
\bar{k}\bar{\sigma} & \equiv k_{p_0+p_1+1} \sigma_{p_0+p_1+1} + \dots + k_p \sigma_p .
\end{aligned} \tag{38}$$



### 3 Quantization of the model

To investigate the quantum properties of the  $p$ -brane we would like to have at our disposal the appropriate classical properties of the original  $p$ -brane. The motion of the  $p$ -brane in the  $\bar{X}^\mu$  variables is the same as described by the original action (13), where all difficulties are hidden in the constraint conditions (33). Finding the solution of the wave equation obeying these constraint conditions is an intricate task in itself, and its solution is yet unknown. As a first step, let us consider the quadratic action under  $X^\mu$  variables, which may be interpreted as an action in the original variables  $X^\mu$ :

$$S = -\frac{T}{2} \int d^{p+1} \xi h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}, \quad (39)$$

where  $h^{\alpha\beta} = \eta^{\alpha\beta}$ ,  $\alpha, \beta = 0, \dots, p$ ;  $g_{\mu\nu} = \eta_{\mu\nu}$ ,  $\mu, \nu = 0, \dots, D-1$ .

The action (39) is invariant respecting the global  $D$ -dimensional Poincaré transformations, but not invariant under local conformal and reparametrization transformations.

The absence of reparametrizations means the absence of the constraints. This allows an easy quantization of the quadratic action.

Consider  $X^\mu(\xi)$  the open  $p$ -brane. Then the solution of the equation of motion (32) is like that of (37), and the density of the energy-momentum tensor

$$\begin{aligned} P_\tau^\mu &= -\frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} = \\ &= -i \sqrt{\frac{2^{p-1} T}{\pi^p}} \sum_{\mathbf{n}} (\alpha_{\mathbf{n}}^\mu e^{-in\tau} - \alpha_{\mathbf{n}}^{*\mu} e^{in\tau}) \prod_{i=1}^p \cos n_i \sigma_i, \\ \alpha_0^\mu &= \frac{1}{\sqrt{2^{p+1} \pi^p T}} p^\mu, \quad \mathbf{n} \in \mathbf{N}^p. \end{aligned} \quad (40)$$

In the light-cone coordinates with assumption that tangent components  $\alpha_{\mathbf{n}}^\pm$  are physical meaningless, like in the string case, we have from the com-

mutation relations

$$[X^\mu(\tau, \sigma), P_\tau^\nu(\tau, \sigma')] = i\eta^{\mu\nu}\delta(\sigma - \sigma') \quad (41)$$

on the quantum level

$$[\alpha_{\mathbf{m}}^i, \alpha_{\mathbf{n}}^j] = n\eta^{ij}\delta_{\mathbf{m}, \mathbf{n}} \quad , \quad (42)$$

where  $\alpha_{\mathbf{n}}^{*\nu} \rightarrow \alpha_{\mathbf{n}}^{+\nu}$ .

The quantum Hamiltonian  $H = \int_0^\pi d^p\sigma (P_\tau^\mu \dot{X}_\mu - \mathcal{L})$  is

$$\begin{aligned} H &= \frac{T}{2} \int_0^\pi (\dot{X}^2 + X_1^2 + \dots + X_p^2) d^p\sigma = \\ &= \alpha_0^2 + \sum_{\mathbf{n}} \alpha_{\mathbf{n}}^+ \alpha_{\mathbf{n}} + \frac{D-p-1}{2} \sum_{\mathbf{n}} n, \quad \mathbf{n} \in \mathbf{N}^p \setminus 0. \end{aligned} \quad (43)$$

As could be expected, the excitations of the linearized model are an ordinary sum of the infinite number of harmonic oscillations described by creating and annihilating operators.

The zero-point energy of the infinite number oscillators (the Casimir energy) diverges, and for correct definition it must be regularized.

Consider the regularization by the contracted Riemann zeta-function:

$$\zeta'_p(s) = \sum_{\mathbf{n}} (n_1^2 + n_2^2 + \dots + n_p^2)^{-s}, \quad \mathbf{n}_i \in \mathbf{N}^p \setminus 0, \quad (44)$$

for which the following properties are known :

$$\zeta'_p(s) = \frac{\pi^p}{\Gamma(s)} \int_0^\infty dt t^{s-1} \sum_{\mathbf{n}} \exp[-\pi(n_1^2 + n_2^2 + \dots + n_p^2)t] \quad (45)$$

and

$$\zeta'_p(s) = \pi^{2s-p/2} \frac{\Gamma(-s+p/2)}{\Gamma(s)} \zeta_p(-s+p/2) \quad . \quad (46)$$

In our case  $s = -\frac{1}{2}$ . According to the definition and the above-mentioned

properties, we can find the first meanings of the  $\zeta'_p(-\frac{1}{2})$ :

$p$	$\zeta'_p(-\frac{1}{2})$	$D_{cr}$
2	0.026	79.623
3	0.053	42.080
4	0.048	46.610
5	0.036	61.603
6	0.249	15.032
7	0.017	128.829
8	0.011	199.398

Then, substituting the quantities  $\zeta'_p(-\frac{1}{2})$  in (43), we obtain the undiverging meanings of the Casimir energy and, correspondingly, good properties of the Hamiltonian  $H$ .

We remember that in the quantum case we have no constraints for this model. But we may impose "by hand" an additional condition  $H|\phi\rangle = 0$ . In this case, we obtain that for the existence of a massless vector, the coefficients at the second term in (43) must equal to one. This condition gives  $D = D_{cr} = 1 + p + 2(\sum_{n>0} n)^{-1}$ . Hence, the ground state of this model is a tachyon.

Now, let us consider  $X^\mu(\xi)$  as a closed  $p$ -brane with the period  $2\pi$  (a toroidal type of the  $p$ -brane). Then, the solution of the equations of motion is like that of (37), and the density of the energy-momentum tensor

$$\begin{aligned}
P_\tau^\mu &= -\frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} = \\
&= -i\sqrt{\frac{2^{p-1}T}{\pi^p}} \sum_{\mathbf{k}} [(\alpha_{\mathbf{k}}^\mu e^{-ik\tau} - \alpha_{\mathbf{k}}^{*\mu} e^{ik\tau})e^{-i\bar{k}\bar{\sigma}} + \\
&\quad + (\beta_{\mathbf{k}}^\mu e^{-ik\tau} - \beta_{\mathbf{k}}^{*\mu} e^{ik\tau})e^{i\bar{k}\bar{\sigma}}], \\
\alpha_0^\mu &= \beta_0^\mu \frac{1}{2\sqrt{2^{p+1}\pi^p T}} p^\mu, \quad \mathbf{k} \in \mathbb{N}^p.
\end{aligned} \tag{47}$$

The left-right symmetry condition gives us the correlation between the coefficients  $\alpha_{\mathbf{k}}^\mu$  and  $\beta_{\mathbf{k}}^\nu$

$$\beta_{\mathbf{k}}^\mu = \alpha_{-\mathbf{k}}^\mu . \quad (48)$$

In this case, from the commutation relations (41) it follows that

$$[\alpha_{\mathbf{m}}^\mu, \alpha_{\mathbf{n}}^{+\nu}] = n\eta^{\mu\nu} \delta_{\mathbf{m},\mathbf{n}} \quad , \quad [\beta_{\mathbf{m}}^\mu, \beta_{\mathbf{n}}^{+\nu}] = n\eta^{\mu\nu} \delta_{\mathbf{m},\mathbf{n}} \quad , \quad (49)$$

$$[\alpha_{\mathbf{m}}^\mu, \beta_{\mathbf{n}}^{+\nu}] = [\alpha_{\mathbf{m}}^\mu, \beta_{\mathbf{n}}^{+\nu}] = 0 . \quad (50)$$

The quantum Hamiltonian

$$H = \frac{T}{2} \int_0^{2\pi} (\dot{X}^2 + X_1^2 + \dots + X_p^2) d^p \sigma = \alpha_0^2 + \beta_0^2 + \frac{1}{2} \sum_{\mathbf{k}} \{\alpha_{\mathbf{k}}^+, \alpha_{\mathbf{k}}\} + \frac{1}{2} \sum_{\mathbf{k}} \{\beta_{\mathbf{k}}^+, \beta_{\mathbf{k}}\} = H_L + H_R , \quad (51)$$

where  $H_L(H_R)$  depends only on  $\alpha_{\mathbf{k}}^\mu(\beta_{\mathbf{k}}^\mu)$  variables and  $\mathbf{k} \in \mathbf{N}^p \setminus 0$ .

In the case of the toroidal  $p$ -brane, we have two different possibilities: (a) to impose a more detailed condition  $H_L|\varphi\rangle = H_R|\varphi\rangle = 0$  or an equivalent  $H|\varphi\rangle = H_L|\varphi\rangle = 0$  ( $H_L|\varphi\rangle = H_R|\varphi\rangle = 0$ ); (b) using the discrete symmetry condition  $X^\mu(\sigma, \tau) = X^\mu(-\sigma, \tau)$  and, consequently, the correlation between  $\alpha_{\mathbf{k}}^\mu$  and  $\beta_{\mathbf{k}}^\nu$  operators, we may impose only one condition  $H|\varphi\rangle = 0$ .

In the first case, we have the same properties for the closed  $p$ -brane as for the open one:

$$H_L = \beta_0^2 + \sum_{\mathbf{k}} \beta_{\mathbf{k}}^+ \beta_{\mathbf{k}} + \frac{D-p-1}{2} \sum_{\mathbf{k}} k , \quad H_L = \alpha_0^2 + \sum_{\mathbf{k}} \alpha_{\mathbf{k}}^+ \alpha_{\mathbf{k}} + \frac{D-p-1}{2} \sum_{\mathbf{k}} k \quad (52)$$

where  $\mathbf{k} \in \mathbf{N}^p \setminus 0$  and, according to the conditions (a), we obtain a tachyon in a ground state and the  $D_{cr}$  corresponding to that in the table for the open  $p$ -brane.

In the second case we may express  $H$  only in the terms of the right (left) operators  $\alpha_{\mathbf{k}}^\mu(\beta_{\mathbf{k}}^\mu)$ :

$$H_L = \alpha_0^2 + \sum_{\mathbf{k} \neq 0} \alpha_{\mathbf{k}}^+ \alpha_{\mathbf{k}} + \frac{D-p-1}{2} \sum_{\mathbf{k} \neq 0} k , \quad (53)$$

where  $k = \sqrt{k_1^2 + \dots + k_p^2}$ .

Using the definition of the ordinary Riemann zeta-function [14]

$$\zeta_p(s) = \sum_{k \neq 0} (k_1^2 + k_2^2 + \dots + k_p^2)^{-s}, \quad (54)$$

with the same properties (45),(46), we may find the first meanings of  $\zeta_p(-\frac{1}{2})$ :

$p$	$\zeta_p(-\frac{1}{2})$	$D_{cr}$
2	-0.229	11.734
3	-0.267	11.491
4	-0.297	11.734
5	-0.325	12.154
6	-0.373	13.362
7	-0.407	12.914
8	-0.462	13.329

Then, substituting the quantities  $\zeta_p(-\frac{1}{2})$  in (53), we find no divergencies of the Hamiltonian  $H$ . In this case, the ground state of the toroidal  $p$ -brane is also a tachyon, and the critical dimension  $D_{cr} = 1 + p - 2(\sum_{k \neq 0} k)^{-1}$ .

## 4 A supersymmetric linearized model

It is of interest to examine the supersymmetric case of the bosonic  $p$ -brane in the GS and NSR approaches. Let us consider the supersymmetric linearized model in the NSR approach. Let  $p = 2$ . Passing over to the  $p \geq 2$  will be simple.

The direct generalization of the linearized model of the bosonic action is

$$S = -\frac{T}{2} \int d^3\xi (\partial_\alpha X^\mu \partial^\alpha X_\mu + i\bar{\psi}^\mu \gamma^\beta \partial_\beta \psi_\mu), \quad (55)$$

where  $\psi^\mu$  is the Majorana spin-vector,  $\{\gamma^\alpha, \gamma^\beta\} = -2\eta^{\alpha\beta}$ . We shall use the basis for  $\gamma^\alpha$ :

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (56)$$

This action is invariant under the transformations

$$\delta X^\mu = i\bar{\eta}\psi^\mu, \quad \delta\psi^\mu = (\gamma^\alpha \partial_\alpha X^\mu)\eta. \quad (57)$$

The equations of motion, which follow from the action of the super  $p$ -brane, are

$$\partial_\alpha \partial^\alpha X^\mu = 0, \quad \partial_2 \psi_1^\mu + (\partial_1 - \partial_0) \psi_2^\mu = 0, \quad (\partial_1 + \partial_0) \psi_1^\mu - \partial_2 \psi_2^\mu = 0. \quad (58)$$

For the variables  $X^\mu$  we may use the same solution as in the bosonic case (37). Let the solutions of the equations of motion for the fermionic part have the following form:

$$\psi_1^\mu(\xi) = \sum_{\mathbf{n}} d_{\mathbf{n}}^{\mu(1)} e^{-i(n_0\tau + n_1\sigma_1 + n_2\sigma_2)}, \quad (59)$$

$$\psi_2^\mu(\xi) = \sum_{\mathbf{n}} d_{\mathbf{n}}^{\mu(2)} e^{-i(n_0\tau - n_1\sigma_1 + n_2\sigma_2)}, \quad (60)$$

where  $\mathbf{n} \in \mathbf{Z}^3$

In this case the equation of motion imposes restrictions on the coefficients  $d_{\mathbf{n}}^{\mu(1)}$  and  $d_{\mathbf{n}}^{\mu(2)}$ :

$$d_{\mathbf{n}}^{\mu(2)} = n_2/(n_0 - n_1) d_{\mathbf{n}}^{\mu(1)} = (n_0 + n_1)/n_2 d_{\mathbf{n}}^{\mu(1)} \quad (61)$$

The Hamiltonian of such system equals to

$$H = \frac{T}{2} \int d^3\xi [\dot{X}^2 + X_1^2 + X_2^2 + \frac{i}{2}(\bar{\psi}^\mu \gamma^0 \dot{\psi}_\mu - \dot{\bar{\psi}}^\mu \gamma^0 \psi_\mu)] \quad (62)$$

In the quantum case, the coefficients  $d_{\mathbf{n}}^{\mu(i)}$  obey the anticommutation relations:

$$\{d_{\mathbf{m}}^{\mu(i)}, d_{\mathbf{n}}^{\nu(j)}\} = \eta^{\mu\nu} \delta_{\mathbf{m}, -\mathbf{n}}. \quad (63)$$

Therefore, the Hamiltonian is the sum of the bosonic and fermionic oscillations:

$$H = \alpha_0^2 + \sum_{\mathbf{n}} [\alpha_{\mathbf{n}}^+ \alpha_{\mathbf{n}}] + \sum_{i, \mathbf{n}} [n d_{\mathbf{n}}^{+(i)} d_{\mathbf{n}}^{(i)}], \quad (64)$$

for which there is no Casimir energy. This is what must be the case with the supersymmetric model.

The initial action does not contain any auxiliary metric on the worldvolume, hence the constraints in the system are absent, too. Like in the bosonic case, we may impose an additional condition  $H = 0$  and consider it in the quantum case as well.

In this case, we find that in the supersymmetric model the condition  $H = 0$  gives us massless ground states and no critical dimensions whatsoever.

## 5 Discussion

In this article we have considered the simplest case of the bosonic and fermionic membranes, when they contain only linear terms in their equations of motion. The general situation is much more complicated.

An essential point of our consideration is imposing additional conditions like  $H = 0$ . But in the case of the linearized model we can consider these conditions as a certain remnant constraint condition like  $L_n = 0$ .

One would remark that  $D_{cr}$  in the bosonic case is not an integer and, consequently, has no physical meaning. Indeed, in all considered cases  $D_{cr} \neq \mathbf{N}$ . But even in the case when  $D_{cr} \in \mathbf{N}$ ,  $D_{cr}$  has no physical meaning. The point is that we cannot pick out physical states among all possible states in the Hilbert space, as we have not enough constraints or the conditions like those and can not obtain the physical sector. On the other hand, the discrete values of the spacetime dimension  $D_{cr}$  imply the existence of the fractal properties of the extended objects. Some of the aspects of these properties are considered in [15].

In the supersymmetric case we have additional possibilities to impose condition, at which the supercurrent  $J^\alpha = K\gamma^\beta\gamma^\alpha\psi^\mu\partial_\beta X_\mu$  vanishes. In this case the condition  $J^\alpha = 0$  is equivalent to six conditions  $\partial_\alpha X^\mu\psi_\mu^i = 0$  or their Fourier transformation  $F_{\mathbf{n}}^{\alpha i} = \int_{-\pi}^{\pi} d^2\sigma e^{i\vec{n}\vec{\sigma}}\partial_\alpha X^\mu\psi_\mu^i$ . The supersymmetric action contains the constraints  $F_{\mathbf{n}}^{\alpha i} = 0$ . We may also express this quantity in the  $\alpha_{\mathbf{n}}, d_{\mathbf{n}}^{(i)}$  variables and consider the quantum case, but this will be also not enough to distinguish the physical sector. Nevertheless, due to the quadratic action we can analytically calculate the partition function and transition amplitude for this model.

The linearized model allows us to separate linear and nonlinear effects in the general (super) $p$ -brane. For instance, in [16], due to the restriction of the constraint condition for the bosonic  $p$ -brane,  $D_{cr}$  has been obtained, whereas the purely linearized model has no critical dimensions. This means that in [16] a nontrivial conformity between the linearized model and the imposed constraint condition was obtained.

We may try to impose sufficient constraint conditions as an additional condition, but in this case a very important question arises: how to conform the solution of the equation of motion with the constraint conditions? We can make it sure that in the bosonic sector the simplest quadratic constraints  $\dot{X}^2 + X_{;1}^2 + \dots + X_{;p}^2 = 0$ ,  $\dot{X}^\mu X_{;i\mu} = 0$ , which are a natural generalization of the string constraints, cannot coexist with the solutions of the linear wave equation of motion for the bosonic  $p$ -brane. Thus, the conformity between the solution of the equation of motion in the linearized model and the additional constraint conditions is nontrivial and of interest in itself.

On the other hand, we may not only use global supersymmetry and vanishing of the supercurrent  $J^\alpha$ , but also the condition of local supersymmetry may be imposed. Indeed, we may use the linearized model of the (super) $p$ -brane with local supersymmetry and try to find the conformity between the solutions and constraints. However, (1) it is not clear how to do it even in a less complicated case without supersymmetry, and (2) this will be not enough to distinguish the physical sector, either.



Thus, we may consider the linearized model an auxiliary model of the (super) $p$ -brane. An important aspect of this consideration is the possibility to separate the physical properties belonging to the linearized model from other properties characteristic of the essentially nonlinear behavior of the relativistic (super) $p$ -brane.

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